Introduction to Integration

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Integration - Overview

A great achievement of classical geometry was to obtain formulas for the areas of triangles, and volumes of spheres and cones. Here we develop a method to calculate the areas and volumes of very general shapes.

This method, called **integration**, is a tool for calculating much more than areas and volumes.

The **integral** is of fundamental importance in statistics, economics, the sciences and engineering.

A variety of applications of integrals will be disucssed in the course.

Definite Integral

The **definite integral** is the key tool in calculus for defining and calculating quantities important to mathematics and science, such as

- areas
- volumes
- lengths of curves paths
- probabilities, and
- the weights of various objects,

just to metion a few.

Definite Integral

The **idea behind the integral** is that we can effectively compute such quantities by **breaking them into small pieces** and then summing the contributions from each piece.

We then consider what happens when more and more, smaller and smaller pieces are taken in the summation process.

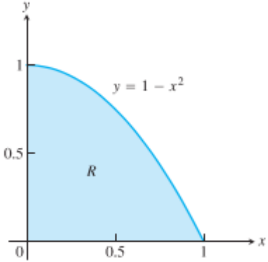
Finally, if the number of terms contributing to the sum approaches infinity and **we take the limit of these sums, the result is a definite integral.**

We shall prove that integrals are connected to antiderivatives, a connection that is one of the most important relationships in calculus.

The basis for formulating definite integrals is the construction of appropriate finite sums.

A method for determining exact area

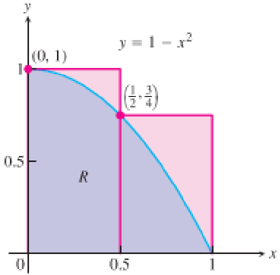
Suppose we want to find the area of the shaded region R that lies above the x-axis, below the graph of $y = 1 - x^2$, and between the vertical lines $x = 0$ and $x = 1$.



Unfortunately, there is no simple geometric formula for calculating the areas of general shapes having curved boundaries like the region R. How, then, can we find the area for R?

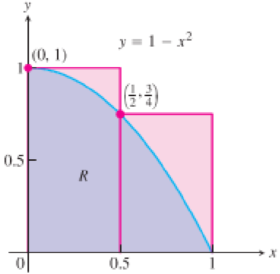
A method for determining exact area

We can approximate the area of R in a simple way. The following figure shows two rectangles that together contain the region R.



The total area of two rectangles approximates the area A of the region R:

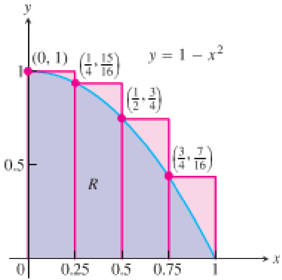
$$A \approx 1.\frac{1}{2} + \frac{3}{4}.\frac{1}{2} = \frac{7}{8} = 0.875.$$



We say that 0.875 is an **upper sum** because it is obtained by taking the height of each rectangle as the maximum (uppermost) value of $f(x)$ for a point x in the base interval of the rectangle.

A method for determining exact area

In the following figure, we improve our estimate by using four thinner rectangles, each of width $1/4$, which taken together contain the region R.



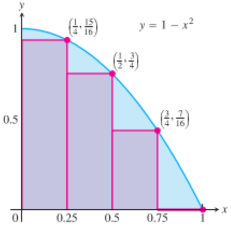
These four rectangles give the approximation

$$A \approx 1.\frac{1}{4} + \frac{15}{16}.\frac{1}{4} + \frac{3}{4}.\frac{1}{4} + \frac{7}{16}.\frac{1}{4} = \frac{25}{32} = 0.78125$$

which is still greater than A since the four rectangles contain R.

A method for determining exact area

Suppose instead we use four rectangles **contained inside the region** R to estimate the area, as in the following figure.



Each rectangle has width $1/4$ as before, but the rectangles are shorter and lie entirely beneath the graph of f. Summing these rectangles with heights equal to the minimum value of $f(x)$ for a point x in each base subinterval gives a **lower sum** approximation to the area,

$$A \approx \frac{15}{16}.\frac{1}{4} + \frac{3}{4}.\frac{1}{4} + \frac{7}{16}.\frac{1}{4} + 0.\frac{1}{4} = \frac{17}{32} = 0.53125.$$

A method for determining exact area

This estimate is smaller than the area A since the rectangles all lie inside of the region R. The true value of A lies somewhere between these lower and upper sums :

$$0.53125 < A < 0.78125.$$

By considering both lower and upper sum approximations we get not only estimates for the area, **but also a bound on the size of the possible error** in these estimates since the true value of the area lies somewhere between them.

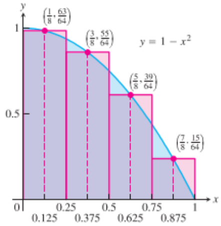
Here the error cannot be greater than the difference

$$0.78125 - 0.53125 = 0.25.$$

The difference between the true (actual) value and an estimate (approximation) of that value is called **error**.

A method for determining exact area

Yet another estimate can be obtained by using rectangles whose heights are the values of f at the midpoints of their bases, as shown in the following figure.



This method of estimation is called the **midpoint rule** for approximating the area.

The midpoint rule gives an estimate that is between a lower sum and an upper sum, but it not quite so clear whether **it overestimates or underestimates the true area.**

A method for determining exact area

In each of our computed sums, the interval $[a, b]$ over which the function f is defined was subdivided into n subintervals of equal width (also called length)

$$\Delta x = \frac{b-a}{n}$$

and f was evaluated at a point in each subinterval:

c_1 in the first subinterval, c_2 in the second subinterval, and so on.

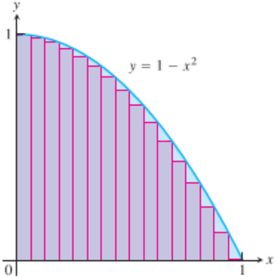
The finite sums then all take the form

$$f(c_1)\Delta x + f(c_2)\Delta x + f(c_3)\Delta x + \cdots + f(c_n)\Delta x.$$

A method for determining exact area

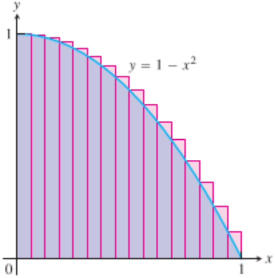
By taking more and more rectangles, with each rectangle thinner than before, it appears that these finite sums give better and better approximations to the true area of the region R.

The following figure shows **a lower sum approximation** for the area of R using 16 rectangles of equal width. The sum of their areas is 0.634765625, which appears close to the true area, but is still smaller since the rectanlges lie inside R.



A method for determining exact area

The following figure shows **an upper sum approximation** using 16 rectangles of equal width. The sum of their areas is 0.697265625, which is somewhat larger than the true area because the rectangles taken together contain R.



The midpoint rule for 16 rectangles gives a total area approximation of 0.6669921875, but it is not immediately clear whether their estimate is larger or smaller than the true area.

A method for determining exact area

The following table shows the values of upper and lower sum approximations to the area of R using up to 1000 rectangles.

Number of subintervals	Lower sum	Midpoint rule	Upper sum
2	.375	.6875	.875
4	.53125	.671875	.78125
16	.634765625	.6669921875	.697265625
50	.6566	.6667	.6766
100	.66165	.666675	.67165
1000	.6661665	.66666675	.6671665

We shall now see how to get an exact value of the areas of regions such as R by taking a limit as the base width of each rectangle goes to zero and the number of rectangles goes to infinity. **We shall develop some techniques to show that the area of R is exactly $2/3$.**

Distance Traveled

Suppose we know the velocity function $v(t)$ of a car moving down a highway, without changing direction, and want to know how far it traveled between times $t = a$ and $t = b$. If we already know an anti-derivative $F(t)$ of $v(t)$ we can find the car's position function $s(t)$ by setting

$$s(t) = F(t) + C.$$

The distance traveled can then be found by calculating the change in position, $s(b) - s(a)$. If the velocity function is determined by recording a speedometer reading at various times on the car, then we have no formula from recording a speedometer reading at various times on the car, then we have no formula from which ti obtain an anti-derivative function for velocity. So what do we do in this situation?

$$\text{distance} = \text{velocity} \times \text{time}$$

and add the result across $[a, b]$.

Distance Traveled

Suppose the subdivided interval looks like with the subintervals all of equal length Δt. Pick a number t_1 in the first interval. If Δt is so small that the velocity barely changes over a short time interval of duration Δt, then the distance traveled in the first time interval is about $v(t_1)\Delta t$.

If t_2 is a number in the second interval, the distance traveled in the second time interval is about $v(t_2)$. The sum of the distances traveled over all the time intervals is

$$\Delta \approx v(t_1)\Delta t + v(t_2)\Delta t + \cdots + v(t_n)\Delta t,$$

where n is the total number of subintervals.

Estimating the Height of a Projectile

Example

The velocity function of a projectile fired straight into the air is

$$f(t) = 160 - 9.8t \quad m/sec.$$

Use the summation technique just described to estimate how far the projectile rises during the first 3 sec. How close do the sums come to the exact figure of $435.9m$?

Solution

We explore the results for different numbers of intervals and different choices of evaluation points. Notice that $f(t)$ is decreasing, so choosing left endpoints gives an upper sum estimate; choosing right endpoints gives a lower sum estimate.

(a) Three subintervals of length 1, with f evaluated at left endpoints giving an upper sum. With f evaluated at $t = 0$, and 2, we have
$D \approx f(t_1)\Delta t + f(t_2)\Delta t + f(t_3)\Delta t$
$= [160 - 9.8(0)](1) + [160 - 9.8(1)](1) + [160 - 9.8(2)](1) = 450.6.$

(b) Three subintervals of length 1, with f evaluated at right endpoints giving a lower sum. With f evaluated at $t = 1, 2$ and 3, we have
$D \approx f(t_1)\Delta t + f(t_2)\Delta t + f(t_3)\Delta t$
$= [160 - 9.8(1)](1) + [160 - 9.8(2)](1) + [160 - 9.8(3)](1) = 421.2.$

(c) With six subintervals of length $1/2$, we get an upper sum using left endpoints: $D \approx 443.25$; a lower sum using right endpoint: $D \approx 428.55$.

Solution (contd...)

These six-interval estimates are somewhat closer than the three-interval estimates. The result improve as the subintervals get shorter.

Add Table page 330 (5.2)

Solution (contd...)

As we can see in Table 5.2, the left-endpoint upper sums approach the true value 435.9 from above, whereas the right-endpoint lower sums approach it from below. The true value lies between these upper and lower sums. The magnitude of the error in the closest entries is 0.23, a small percentage of the true value.

$$\begin{aligned}
\texttt{Error magnitude} &= |\texttt{true value} - \texttt{calculated value}| \\
&= |135.9 - 435.67| = 0.23 \\
\texttt{Error percentage} &= \frac{0.23}{435.9} \approx 0.05\%
\end{aligned}$$

It would be reasonable to conclude from the table's last entries that the projectile rose about 436 m during its first 3 sec of flight.

Displacement Versus Distance Traveled

If a body with position function $s(t)$ moves along a coordinate line without changing direction, we can calculate the total distance it travels from $t = \alpha$ to $t = b$ by summing the distance traveled over small intervals, as in Example 2. If the body changes direction one or more times during the trip, then we need to use the body's speed $|v(t)|$, which is the absolute value of its velocity function, $v(t)$, to fin the total distance traveled. Using the velocity itself, as in Example 2, only gives an estimate to the body's **displacement**, $s(b) - s(a)$, the difference between its initials and final positions.

Displacement Versus Distance Traveled

To see why, partition the time interval if $[a, b]$ into small enough equal subintervals Δt so that the body's velocity does not change very much from time t_{k-1} to t_k. Then $v(t_k)$ gives a good approximation of the velocity throughout the interval. Accordingly, the change in the body's position coordinate during the time interval is about

$$v(t_k)\Delta t.$$

The change is positive if $v(t_k)$ is positive and negative if $v(t_k)$ is negative. In either case, the distance traveled during the subinterval is about

$$|v(t_k)|\Delta t.$$

The **total distance traveled** is approximately the sum

$$|v(t_1)|\Delta t + |v(t_2)|\Delta t + \ldots + |v(t_n)|\Delta t.$$

Displacement Versus Distance Traveled

Add figure 5.5 on page 331

Figure 5.5 (a) The average value of $f(x) = c$ on $[a, b]$ is the area of the rectangle divided by $b - a$.(b) The average value of $g(x)$ on $[a, b]$ is the area beneath its graph divided by $a - b$.

Average Value of a Nonnegative Function

The average value of collection of n numbers $x_1, x_2, \ldots, x_n$ is obtained by adding them together and dividing by n. But what is the average values of a continuous function f on an interval $[a, b]$? Such a function can assume infinitely many values. For example, the temperature at a certain location in a town is a continuous function that goes up and down each day. What it mean to say the average temperature in the town over the course of a day is 73 degrees?

When a function is constant, this question is easy to answer. A function with constant value cn an interval $[a, b]$ has average value c is positive, its graph over $[a, b]$ gives a rectangle of height c. The average value of the function can then be interpreted geometrically as the area of this rectangle divided by its width $b - a$ (Figure 5.5 a).

Average Value of a Nonnegative Function

What if we want to find the average value of a non constant function, such as the function g if Figure 5.5b? We can think of this graph as a snapshot of the height of some water that is sloshing around in tank, between enclosing wall at $x = a$ and $x = b$. As the water moves, its height over each point changes, but its average height remains the same. To get the average height of the water, we let it settle down until it is level and its height is constant. The resulting height c equals the area under the graph of g divided by $b - a$. We are led to define the average value of nonnegative function on an interval $[a, b]$ to be the area under its graph divided by $b - a$. For this definition to be valid, we need a precise understanding of what is meant by the area under a graph. This will be obtained in Section 5.3, but for now we look at two simple examples.

Example

What is the average value of the function $f(x) = 3x$ *on the interval*$[0, 2]$?

Solution The average equals the area under the graph divided by the width of the interval. In this case we do not need finite approximation to estimate the area of the region under the graph: a triangle of height 6 and base 2 has area 6 (Figure 5.6.) The width of the interval is $b - a = 2 - 0 = 2$. The average value of the function is $6/2 = 3$.

The Average Value of $\sin x$

Example

Estimate the average value of the function $f(x) = \sin x$ *on the interval* $[0, \pi]$

Solution

Looking at the graph of $\sin x$ between 0 and π in Figure 5.7, we can see that its average height is somewhere between 0 and 1.

To find the average we need to calculate the area A under the graph and then devide this area by the length of the interval, $\pi - 0 = \pi$.

page 332 figure 5.7

We do not have a simple way to determine the area, so we approximate it with finite sums. To get an upper sum estimate, we add the areas of four rectangles of equal width $\pi/4$ that together contain the region beneath the graph of $y = \sin x$ and above the $x - axis$ on $[0, \pi]$.

We choose the heights of the rectangles to be the largest value of $\sin x$ on each subinterval. Over a particular subinterval, this largest value may occur at the e left endpoint, the right endpoint, or somewhere between them.

We evaluate $\sin x$ at this point to get the height of the rectangle for an upper sum. The sum of the rectangle areas then estimates the toatal area Figure 5.7a):

$$\begin{aligned} A &\approx \left(\sin \frac{\pi}{4}\right).\frac{\pi}{4} + \left(\sin \frac{\pi}{2}\right).\frac{\pi}{4} + \left(\sin \frac{\pi}{2}\right).\frac{\pi}{4} + \left(\sin \frac{3\pi}{4}\right).\frac{\pi}{4} \\ &= \left(\frac{1}{\sqrt{2}} + 1 + 1 + \frac{1}{\sqrt{2}}\right).\frac{\pi}{4} \approx (3.42).\frac{\pi}{4} \approx 2.69 \end{aligned}$$

To estimate the average value of $\sin x$ we divided the estimated area by π and obtain the approximation $2.69/\pi \approx 0.86$.
If we use eight rectangles of equal width $\pi/8$ all lying above the graph of $y = \sin x$(figure 5.7b), we get the area estimate

$$\begin{aligned} A &\approx \left(\sin\frac{\pi}{8} + \sin\frac{\pi}{4} + \sin\frac{3\pi}{8} + \sin\frac{\pi}{2} + \sin\frac{\pi}{2} + \sin\frac{5\pi}{8} + \sin\frac{3\pi}{4} + \sin\frac{7\pi}{8}\right) \\ &\approx (.38 + .71 + .92 + 1 + 1 + .92 + .71 + .38).\frac{\pi}{8} = (6.02).\frac{\pi}{8} \\ &\approx 2.365. \end{aligned}$$

Dividing this result by the length π of the interval gives a more accurate estimate of 0.753 for the average. Since we used an upper sum approximate the area, this estimate is still greater than the actual average value of $\sin x$ over $[0,\pi]$. If we use more and more rectangles, with each rectangle getting thinner and thinner, we get closer to the true average value. Using the techniques of Section 5.3, we will show that the true average value is $2/\pi \approx 0.64$.

As before,we could just as well have used rectangles lying under the graph of $y=\sin x$ and calculated a lower sum approximation, or we could have used the midpoint rule. In section 5.3, we will see that it doesn't matter whether our approximating rectangles. are chosen to give upper sums, lower sums, or a sum in between. In each case, the approximations are close to the true area if all the rectangles are sufficiently thin.

Summary

The area under the graph of a positive function, the distance traveled by a moving object that doesn't change direction, and the average value of a nonnegative function over an interval can all be approximated by finite sums.

First we subdivide the interval into subintervals,treating the appropriate function f as if it were constant over each particular subinterval. Then we multiply the width of each subinterval by the value of f at some point within it, and add these products together.

If the interval [a,b] is subdivided into n subintervals of equal widths $\Delta x = (b-a)/n$, and if $f(c_k)$ is the value of f at the chosen point c_k in the kth subinterval, this process gives a finite sum of the form

$$f(c_1)\Delta x + f(c_2)\Delta x + f(c_3)\Delta x + \cdots + f(c_n)\Delta x.$$

Summary

The choices for the c_k could maximize or minimize the value of f in the kth subinterval,or give some value in between. The true value lies somewhere between the approximations given by upper sums and lower sums.

The finite sum approximations we looked at improved as we took more subintervals of thinner width.

Exercise (Area)

In Exercise 1-4 use finite approximations to estimate the area under the graph of the function using

a. *a lower sum with two rectangles of equal width.*
b. *a lower sum with four rectangles of equal width.*
c. *an upper sum with two rectangles of equal width.*
d. *an upper sum with four rectangles of equal width.*

1. $f(x) = x^2$ *between* $x = 0$ *and* $x = 1$.
2. $f(x) = x^3$ *between* $x = 0$ *and* $x = 1$.
3. $f(x) = 1/x$ *between* $x = 1$ *and* $x = 5$.
4. $f(x) = 4 - x^2$ *between* $x = -2$ *and* $x = 2$.

Exercise

Using rectangles whose height is given by the value of the function at the midpoint of the rectangle's base(the midpoint rule) estimate the area under the graphs of the following functions, using first two and then four rectangles.

1. $f(x) = x^2$ *between* $x = 0$ *and* $x = 1$.
2. $f(x) = x^3$ *between* $x = 0$ *and* $x = 1$.
3. $f(x) = 1/x$ *between* $x = 1$ *and* $x = 5$.
4. $f(x) = 4 - x^2$ *between* $x = -2$ *and* $x = 2$.

Exercise (Distance Traveled)

The accompanying table shows the velocity of a model train engine moving along a track for 10sec. Estimate the distance traveled by the engine using 10 subintervals of length 1 with

a. *left-endpoint values.*

b. *right-endpoint values.*

Time (sec)	Velocity (in./sec)	Time (sec)	Velocity (in./sec)
0	0	6	11
1	12	7	6
2	22	8	2
3	10	9	6
4	5	10	0
5	13		

Exercise (Distance traveled upstream)

You are sitting on the bank of a tidal river watching the incoming tide carry a bottle upstream. You record the velocity of the flow every 5 minutes for an hour, with the results shown in the accompanying table. About how far upstream did the bottle travel during that hour? Find an estimate using 12 subintervals of length 5 with

a. *left-endpoint values.*

b. *right-endpoint values.*

Time (min)	Velocity (m/sec)	Time (min)	Velocity (m/sec)
0	1	35	1.2
5	1.2	40	1.0
10	1.7	45	1.8
15	2.0	50	1.5
20	1.8	55	1.2
25	1.6	60	0
30	1.4		

Exercise

Length of a road *You and a companion are about to drive a twisty stretch of dirt road in a car whose speedometer works but whose odometer(mileage counter) is broken.To find out how long this particular stretch of road is, you record the car's velocity at 10-sec intervals,with the results shown in the accompanying table. Estimate the length of the road using*

a. *left-endpoint values.*

b. *right-endpoint values.*

Time (sec)	Velocity (converted to ft/sec) (30 mi/h=44 ft/sec)	Time (sec)	Velocity (converted to ft/sec) (30 mi/h=44 ft/sec)
0	0	70	15
10	44	80	22
20	15	90	35
30	35	100	44
40	30	110	30
50	44	120	35
60	35		

Exercise (Distance from velocity data)

The accompanying table gives data for the velocity of a vintage sports car accelerating from 0 to 142mi/h in 36 sec (10 thousands of an hour).

Time (h)	Velocity (mi/h)	Time (h)	Velocity (mi/h)
0.0	0	0.006	116
0.001	40	0.007	125
0.002	62	0.008	132
0.003	82	0.009	137
0.004	96	0.010	142
0.005	108		

Add picture on page 334

a. *Use rectangles to estimate how far the car traveled during the 36 sec it took to reach 142mi/h.*

b. *Roughly how many seconds did it take the car to reach the halfway point? About how fast was the car going then?*

Exercise (Free fall with air resistance)

An object is dropped straight down from a helicopter. The object falls faster and faster but its acceleration(rate of change of its velocity) decreases over time because of air resistance.The acceleration is measured in ft/sec^2 and recorded every second after the drop for 5 sec, as shown

t	0	1	2	3	4	5
α	32.00	19.41	11.77	7.14	4.33	2.63

a. *Find an upper estimate for the speed when* $t = 5$.

b. *Find a lower estimate for the speed when* $t = 5$.

c. *Find an upper estimate for the distance fallen when* $t = 3$.

Exercise (Distance traveled by a projectile)

An object is shot straight upward from sea level with an initial velocity of 400 ft/sec.

a. *Assuming that gravity is the only force acting on the object, give an upper estimate for its velocity after 5 sec have elapsed. Use $g = 32$ ft/sec^2 for the gravitational acceleration.*

b. *Find a lower estimate for the height attained after 5 sec.*

Exercise (Average Value of a Function)

In Exercises 15-18, use a finite sum to estimate the average value of f on the given interval by partitioning the interval into four subintervals of equal length and evaluating f at the subinterval midpoints.

15. $f(x) = x^3$ *on* $[0, 2]$
16. $f(x) = 1/x$ *on* $[1, 9]$
17. $f(t) = (1/2) + \sin^2 \pi t$ *on* $[0, 2]$
18. $f(t) = 1 - (\cos \frac{\pi t}{4})^4$ *on* $[0, 4]$

Exercise

[Water pollution] Oil is leaking out of a tanker damaged at sea.The damage to the tanker is worsening as evidenced by the increased leakage each hour,recorded in the following table.

Time(h)	0	1	2	3	4	5	6	7	8
Leakage(gal/h)	50	70	97	136	190	265	369	516	720

a. *Give an upper and a lower estimate of the total quantity of oil that has escaped after 5 hours.*
b. *Repeat part(a)for the quantity of oil that has escaped after 8 hours.*
c. *The tanker continues to leak 720 gal/h after the first 8 hours. If the tanker originally contained 25,000 gal of oil,approximately how many more hours will elapse in the worst case before all the oil has spilled?In the best case?*

Exercise (Air pollution)

A power plant generates electricity by burning oil.Pollutants produced as a result of the burning process are removed by scrubbers in the smokestacks. Over time,the scrubbers become less efficient and eventually they must be replaced when the amount of pollution released exceeds government standards.Measurements are taken at the end of each month determining the rate at which pollutants are released into the atmosphere, recorded as follows.

Month(h)	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Pollutant Release rate (tons/day)	0.20	0.25	0.27	0.34	0.45	0.52	0.63	0.70	0.81	0.85	0.89	0.95

a. *Assuming a 30-day month and that new scrubbers allow only 0.05 ton/day released,give an upper estimate of the total tonnage of pollutants released by the end of June.What is a lower estimate?*

b. *In the best case,approximately when will a total of 125 tons of pollutants have been released into the atmosphere?*

Exercise (Area of a Circle)

Inscribe a regular n-sided polygon inside a circle of radius 1 and compute the area of the polygon for the following values of n:

a. 4 (square) b. 8 (octagon) c. 16

d. Compare the areas in parts (a),(b), and (c) with the area of the circle.

a. *Inscribe a regular n-sided polygon inside a circle of radius 1 and compute the area of one of the n congruent triangles formed by drawing radii to the vertices of the polygon.*
b. *Compute the limit of the area of the inscribed polygon as $n \longrightarrow \infty$.*
c. *Repeat the computations in parts (a) and (b) for a circle of radius r.*

Limits of Finite Sums

The finite sum approximations became more accurate as the number of terms increased and the subinterval widths (lengths) narrowed.

The following example shows how to calculate a limiting value as the widths of the subintervals go to zero and their number grows to infinity.

Example

Find the limiting value of lower sum approximations to the area of the region R below the graph of

$$y = 1 - x^2$$

and above the interval $[0, 1]$ on the x-axis using equal-width rectangles whose widths approach zero and whose number approaches infinity.

Limits of Finite Sums

We compute a lower sum approximation using n rectangles of equal width $\Delta x = (1-0)/n$, and then we see what happens as $n \to \infty$.

We start by subdividing $[0,1]$ into n equal width subintervals.

Each subinterval has width $1/n$.

The function's smallest value in a subinterval occurs at the subinterval's right endpoint. So a lower sum is

$$\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\left(\frac{1}{n}\right) = 1 - \frac{2n^3+3n^2+n}{6n^3}.$$

We have obtained an expression for the lower sum that holds for any n.

Limits of Finite Sums

Taking the limit of this expression as $n \to \infty$, we see that the lower sums converge as the number of subintervals increases and the subintervals widths approach zero :

$$\lim_{n\to\infty}\left(1-\frac{2n^3+3n^2+n}{6n^3}\right)=1-\frac{2}{6}=\frac{2}{3}.$$

The lower sum approximations converge to $2/3$.

A similar calculation shows that the upper sum approximations also converge to $2/3$.

Limits of Finite Sums

Any finite sum approximation

$$\sum_{k=1}^{n} f(c_k)(1/n)$$

also converges to the same value, $2/3$.

This is because it is possible to show that any finite sum approximation is trapped between the lower and upper sum approximations.

For this reason we are led to define the area of the region R as this limiting value.

Riemann Sums

The theory of limits of finite approximations was made precise by the German mathematician Bernhard Riemann.

We now introduce the notion of a Riemann sum. A Riemann sum is an approximation that takes the form $\sum f(x)\Delta x$.

Georg Friedrich Bernhard Riemann (1826-1866)

Bernard Riemann

Bernard Riemann was born near Hanover, Germany. To please his father, he enrolled (1846) at the University of Göttingen as a student of theology and philosophy, but soon switched to mathematics.

He interrupted his studies at Göttingen to study at Berlin under C.G.J. Jacobi, P.G.J. Dirichlet, and F.G. Eisenstein, but returned to Göttingen in 1849 to complete his thesis under Gauss.

His thesis dealt with what are now called "Riemann Surfaces". Gauss was so enthusiastic about Riemann's work that he arranged for him to become a **privatdozent*** at Göttingen in 1854.

***Privatdozent** (for men) or Privatdozentin (for women) is an academic title conferred at some European universities, especially in German-speaking countries, to someone who holds certain formal qualifications that denote an ability to teach a designated subject at university level. In its current usage, the title indicates that the holder has permission to teach and examine independently without being a professor. The title is not necessarily connected to a salaried position, but may entail a nominal obligation to teach.

Bernard Riemann

On admission as a privatdozent, Riemann was required to prove himself by delivering a probationary lecture before the entire faculty. As tradition dicated, he submitted three topics, the first two of which he was well prepared to discuss.

To Riemann's surprise, Gauss chose that he should lecture on the third topic : "On the hypotheses that underlie the foundations of geometry". After its publication, this lecture had a profound effect on modern geometry.

Despite the fact that Riemann contracted tubercluosis and died at the age of 39, he made major contributions in many areas: the foundation of geometry, number theory, real and complex analysis, topology, and mathematical physics.

Riemann Sums

We begin with an arbitrary function f (not necessarily continuous) defined on a closed interval $[a, b]$.

The function f may have negative as well as positive values.

We subdivide the interval $[a, b]$ into subintervals, not necessarily of equal widths.

Starting with 'a' and ending 'b', we choose $n - 1$ points $\{x_1, x_2, \ldots, x_{n-1}\}$ between a and b such that

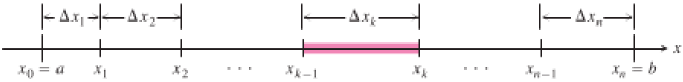
$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < b = x_n.$$

The set $P = \{x_0, x_1, \ldots, x_n\}$ is called a **partition** of $[a, b]$.

The partition P divides $[a, b]$ into n closed subintervals

$$[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n].$$

The width of the kth subinterval $[x_{k-1}, x_k]$ is denoted by Δx_k.



Riemann Sums

In each subinterval we select some point. The point chosen in the kth subinterval $[x_{k-1}, x_k]$ is called c_k.

Then on each subinterval we stand a vertical rectangle that streches from the x-axis to touch the curve at $(c_k, f(c_k))$.

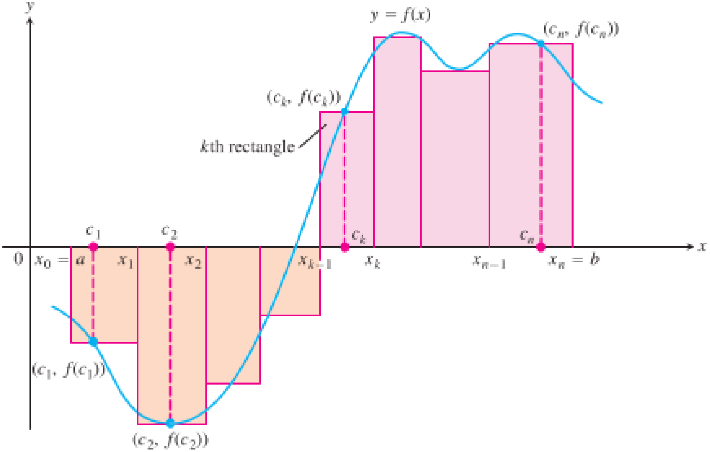
These rectangles can be above or below the x-axis, depending on whether $f(c_k)$ is positive or negative, or on it if $f(c_k) = 0$.

On each subinterval we form the product $f(c_k).\Delta x_k$. This product is positive, negative or zero, depending on the sign of $f(c_k)$.

When $f(c_k) > 0$, the product $f(c_k).\Delta x_k$ is the area of a rectangle with height $f(c_k)$ and width Δx_k.

When $f(c_k) < 0$, the product $f(c_k).\Delta x_k$ is a negative number, the negative of the area of a rectangle of width Δx_k that drops from the x-axis to the negative number $f(c_k)$.

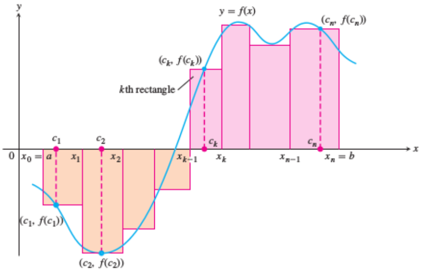
Riemann Sums



Finally we sum all these products to get

$$S_P = \sum_{k=1}^{n} f(c_k)\Delta x_k.$$

The sum S_P is called a **Riemann sum for f on the interval $[a, b]$**. **There are many such sums, depending on the partition P we choose, and the choices of the points c_k in the subintervals.**

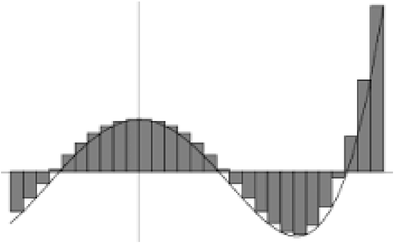


Upper Riemann Sum

If we choose c_k to be maximum value of f on $[x_{k-1}, x_k]$, then

$$S_P = \sum_{k=1}^{n} f(c_k)\Delta x_k$$

gives a sum, called **upper Riemann sum**.

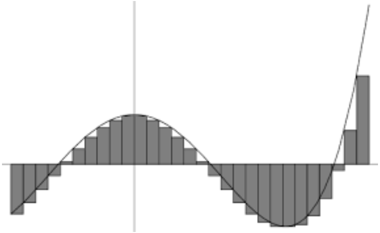


Lower Riemann Sum

If we choose c_k to be minimum value of f on $[x_{k-1}, x_k]$, then

$$S_P = \sum_{k=1}^{n} f(c_k)\Delta x_k$$

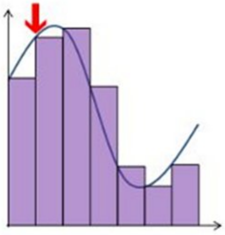
gives a sum, called **lower Riemann sum**.



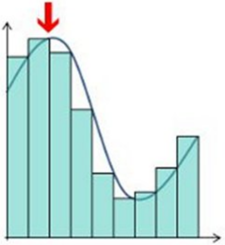
Note that for a given partition P, any Riemann sum is no more than the upper sum of that partition and no less than the lower sum.

Different choices for c_k's

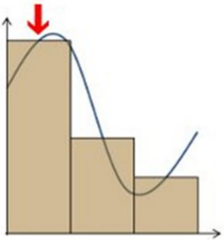
If we choose c_k as left-hand endpoint, right-hand endpoint, and midpoint of the kth subinterval, we shall have different Riemann sums, as shown in the following figures.



The left side of the rectangle



The right side of the rectangle



The middle of the base of the rectangle

Norm of a partition

When a partition has subintervals of varying widths, we can ensure they are all thin by controlling the width of a widest (longest) subinterval.

We define the **norm** of a partition P, written

$$\|P\| \quad \text{(called norm of } P\text{)}$$

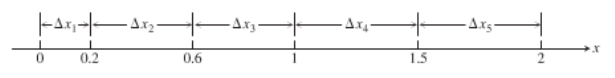
to be the largest of all the subintervals widths.

If $\|P\|$ is a small number, then all of the subintervals in the partition P have a small width.

Norm of a partition

Example

The set $P = \{0, 0.2, 0.6, 1, 1.5, 2\}$ is a partition of $[0, 2]$.



- There are five subintervals of P (6 elements in P).
- The longest subinterval length is 0.5, so the norm of the partition is

$$\|P\| = 0.5.$$

- In this example, there are two subintervals of this length.

Riemann Sums

Many partitions have the same norm, so the partition is **not** a function of the norm.

The points c_k selected from subintervals $[x_{k-1}, x_k]$ are called **tags**.

A set of ordered pairs

$$\left\{[x_{k-1}, x_k], c_k\right\}_{k=1}^{n}$$

of subintervals and corresponding tags is called a **tagged partition** of the interval of $[a, b]$.

Since each tag can be chosed in infinitely many ways, each partition can be tagged in infinitely many ways.

Riemann Sums

The Riemann sum

$$S_P = \sum_{k=1}^{n} f(c_k)\Delta x_k$$

is the sum of the areas of n rectangles whose bases are the subintervals

$$[x_{k-1}, x_k]$$

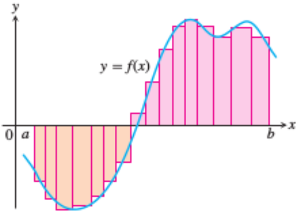
and whose heights are

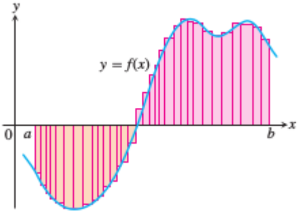
$$f(c_k).$$

Rectangles with thinner bases = Partitions with smaller norms

Any Riemann sum associated with a partition of a closed interval $[a, b]$ defines rectangles that approximate the region between the graph of a function f and the x-axis.

Partitions with norm approaching zero lead to collections of rectangles that approximate this region with increasing accuracy.





Definition of the Definite Integral

The definition of the definite integral is based on the idea that for certain functions, as the norm of the partitions of $[a, b]$ approaches zero, the values of the corresponding Riemann sums approach a limiting value J.

What does this convergence mean ?

A Riemann sum will be close to the number J provided that the norm of its partition is sufficiently small (so that all of its subintervals have thin enough widths).

Definition (Definite integral of f over $[a, b]$)

Let $f(x)$ be a function defined on a closed interval $[a, b]$. We say that a number J is the definite integral of f over $[a, b]$ *and that J is the* limit *of the Riemann sums*

$$\sum_{k=1}^{n} f(c_k)\Delta x_k$$

if the following condition is satisfied:
Given any number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that for every partition

$$P = \{x_0, x_1, \ldots, x_n\}$$

of $[a, b]$ with $\|P\| < \delta$ and any choice of c_k in $[x_{k-1}, x_k]$, we have

$$\left| \sum_{k=1}^{n} f(c_k)\Delta x_k - J \right| < \varepsilon.$$

Definition of the Definite Integral

If we use only the definition in order to show that a function f is (Riemann) integrable we must

- know (or guess correctly) the value J of the integral, and
- construct a δ that will suffice for an arbitrary $\varepsilon > 0$.

The determination of J is sometimes done by calculating Riemann sums and guessing what J must be. The determination of δ is likely to be difficult.

Integrable Function

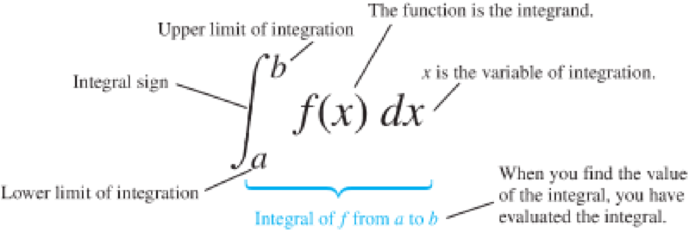
A small positive number ε that specifies how close to J the Riemann sum must be, and a second positive number δ that specifies how small the norm of a partition must be in order for that to happen.

The number J in the definition of the definite integral **is denoted by**

$$\int_a^b f(x)\,dx$$

which is read as "the integral from a to b of f of x dee x" or sometimes as "the integral from a to b of, f of x with respect to x."

Integrable Function



When the definition is satisfied, we say the Reimann sums of f on $[a, b]$ **converge** to the definite integral

$$J = \int_a^b f(x)\ dx$$

and that f is **(Riemann) integrable** over $[a, b]$.

Existence of Definite Integral

We have **many choices for a partition P** with norm going to zero, and **many choices of points c_k** for each partition.

The definite integral exists when we always get the same limit J, **no matter what choices are made.**

When the limit exists we write it as the definite integral

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k = J = \int_a^b f(x)\, dx.$$

It is sometimes said that the integral J is "the limit" of the Riemann sums S_P as the norm

$$\|P\| \to 0.$$

However, since S_P is not a funcion of $\|P\|$, this limit is not of the type that you have studied before.

Existence of Definite Integral

When each partition has n equal subintervals, each of width

$$\Delta x = \frac{b-a}{n}$$

we will also write

$$\lim_{n \to \infty} \sum_{k=1}^{n} f(c_k)\Delta x = J = \int_a^b f(x)\, dx.$$

The limit is always taken as the norm of the partitions approaches zero and the number of subintervals goes to infinity.

Value of integral is unique

Theorem

If f is (Riemann) integrable on $[a, b]$, then the value of the integral is uniquely determined.

Proof.

Suppose that J_1 and J_2 both satisfy the definition. Let $\varepsilon > 0$. Then there exists $\delta_1 > 0$ such that for every partition

$$P_1 = \{x_0, x_1, \ldots, x_n\}$$

of $[a, b]$ with $\|P_1\| < \delta_1$ and any choice of c_k in $[x_{k-1}, x_k]$, we have

$$\left| \sum_{k=1}^{n} f(c_k)\Delta x_k - J_1 \right| < \varepsilon/2.$$

□

Value of integral is unique

contd..

Also there exists $\delta_2 > 0$ such that for every partition

$$P_2 = \{y_0, y_1, \dots, y_m\}$$

of $[a, b]$ with $\|P_2\| < \delta_2$ and any choice of d_j in $[y_{j-1}, y_j]$, we have

$$\Big|\sum_{j=1}^{m} f(d_k)\Delta y_k - J_2\Big| < \varepsilon/2.$$

□

Value of integral is unique

contd..

Let $\delta = \min\{\delta_1, \delta_2\}$ and let P be any partition with $\|P\| < \delta$. Since both $\|P\| < \delta_1$ and $\|P\| < \delta_2$, then any choice of e_i in $[z_{i-1}, z_i]$,

$$\left|\sum_{i=1}^{\ell} f(e_i)\Delta z_i - J_1\right| < \varepsilon/2 \quad \text{and} \quad \left|\sum_{i=1}^{\ell} f(e_i)\Delta z_i - J_2\right| < \varepsilon/2.$$

Hence from the triangle inequality,

$$|J_1 - J_2| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $J_1 = J_2$. □

The proof of above theorem is not included in the course.

Dummy Variable

The value of the definite integral of a function over any particular interval **depends on the function, not on the letter we choose** to represent its independent variable.

If we decide to use t or u instead of x, we simply write the integral as

$$\int_a^b f(t)\ dt \quad \text{or} \quad \int_a^b f(u)\ du \quad \text{instead of} \quad \int_a^b f(x)\ dx.$$

Now matter how we write the integral, it is still the same number that is defined as a limit of Riemann sums.

Since it does not matter what letter we use, the variable of integration is called a **dummy variable** representing the real numbers in the closed interval $[a, b]$.

Rectangles for Riemann Sums

Exercises

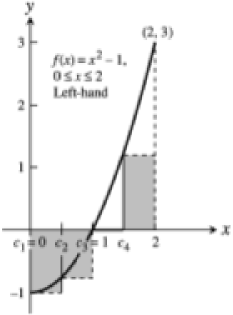
1. *Find the norm of the partition* $P = \{-2, -1.6, -0.5, 0, 0.8, 1\}$.
2. *In the following exercises, graph each function* $f(x)$ *over the given interval.*
   (a) $f(x) = x^2 - 1, \quad [0, 2]$
   (b) $f(x) = -x^2, \quad [0, 1]$
   (c) $f(x) = \sin x + 1, \quad [-\pi, \pi]$.

   Partition the interval into four subintervals of equal length. Then add to your sketch the rectangles associated with the Riemann sum $\sum_{k=1}^{n} f(c_k)\Delta x_k$ *given that* c_k *is the*
   - *left-hand endpoint,*
   - *right-hand endpoint,*
   - *midpoint of the* k*th subinterval. (Make a separate sketch for each set of rectangles.)*

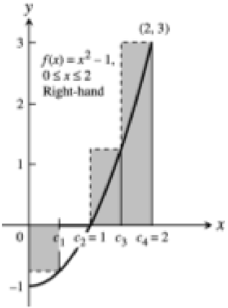
Solution for finding norm of the partition

1. The largest of all the subintervals widths is $\|P\| = 1.1$.

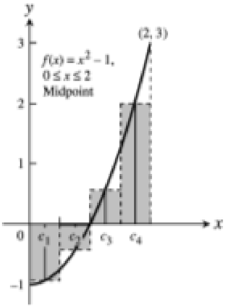
2. **Left-hand point**



2. **Right-hand point**



2. **Midpoint**



Integrable and Nonintegrable Functions

Not every function defined over the closed interval $[a, b]$ is integrable there, even if the function is bounded.

The Riemann sums for some functions
may not converge to the same limiting value, or to any value at all.

Advanced mathematical analysis is required to test whether or not a function defined over $[a, b]$ is integrable.

But fortunately, most functions that commonly occur in applications are **continuous** or **piecewise-continuous** functions.

Piecewise-continuous function is the one having no more than a finite number of jump discontinuities on $[a, b]$.

Integrability of Continuous Functions

Since there are so many choices to be made in taking a limit of Riemann sums, it might seem difficult to show that such a limit exits.

It turns out, however, that no matter what choices are made, the Riemann sums associated with a continuous function converge to the same limit.

The following result tells that every continuous function over $[a, b]$ is integrable over this interval, and so is every piecewise-continuous function.

Theorem (Integrability of Continuous Functions)

If a function f is continuous or piecewise-continuous over the interval $[a, b]$, then the definite integral

$$\int_a^b f(x)\, dx$$

exists and f is integrable over $[a, b]$.

The main idea behind the proof of the above theorem is that the upper and lower sums converge to the same value when

$$\|P\| \to 0.$$

All other Riemann sums lie between the upper and lower sums and have the same limit as well.

Therefore, the number J in the definition of the definite integral exists, and the continuous function f is integrable over $[a, b]$.

Continuous functions are integrable

The theorem says nothing about how to calculate definite integrals.

A method of calculation will be developed through a connection to the process of taking anti-derivatives (Fundamental Theorem of Calculus, Parts 1 and 2).

Functions that are not continuous may or may not be integrable. The following function is discontinuous function but integrable.

Exercise

1. *Let* $f(x) = 1$ *for* $x = \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$, *and* $f(x) = 0$ *elsewhere on* $[0, 1]$. *Then using* ε-δ *definition, show that* f *is Riemann integrable and*

$$\int_0^1 f(x)dx = 0.$$

[Hint : There are four points where f *is not* 0*, each of which can belong to two subintervals. Only these terms will make a nonzero contribution to the Riemann sum.]*

Solution

There are four points where f is not 0, each of which can belong to at most two subintervals.

In the Riemann sum

$$\sum_{k=1}^{n} f(c_k)\Delta x_k,$$

only the subintervals containing the points $\left\{\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\right\}$ will make nonzero contributions to the Riemann sum.

For any partition $P = \{x_0, x_1, \ldots, x_n\}$, we have $\Delta x_k \leq \|P\|$, for each $k = 1, 2, \ldots, n$. Hence

$$\sum_{k=1}^{n} f(c_k)\Delta x_k \leq 8\|P\|.$$

Solution (contd...)

Let $\varepsilon > 0$ be given.

If we choose $\delta < \frac{\varepsilon}{8}$, then for every partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[0, 1]$ with $\|P\| < \delta$ and any choice of c_k in $[x_{k-1}, x_k]$, we have

$$\left| \sum_{k=1}^{n} f(c_k)\Delta x_k - 0 \right| \leq 8\|P\| < 8\delta < \varepsilon.$$

Exercise

Let g be a function defined on $[a, b]$. Suppose that g assumes non-zero values only on a finte set M (that is, g takes 0 on the complement of M in $[a, b]$). Then using ε-δ definition, show that g is Riemann integrable and

$$\int_a^b g(x)dx = 0.$$

Computing integral by limit of Riemann sums

Let f be a continous function on $[a, b]$.

Then f is (Riemann) integrable on $[a, b]$.

The following theorem is helpful to find the integral of f over $[a, b]$.

Theorem

Let f be integrable over $[a, b]$. If $\{P_n\}$ is a sequence of partitions on $[a, b]$ such that $\|P_n\| \to 0$ then

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} S_{P_n}.$$

The statement of above theorem has to be mentioned when you are asked to compute integral of a function over an interval, by using limit of Riemann sums.

Nonintegrable Functions

For integrability to fail, a function needs to be sufficiently discontinuous that the region between its graph and the x-axis, cannot be approximated well by increasingly thin rectangles.

The next example shows a funtion that is not integrable over a closed integral.

Example

The function

$$f(x) = \begin{cases} 1, & \text{if } \; x \text{ is rational} \\ 0, & \text{if } \; x \text{ is irrational} \end{cases}$$

has no Riemann integral over $[0, 1]$.

Nonintegrable Functions

- Between any two numbers there is both a rational number and an irrational number.
- The function f jumps up and down too erratically over $[0, 1]$.
- If we pick a partition P of $[0, 1]$ and choose c_k to be the point giving the maximum value for f on $[x_{k-1}, x_k]$, then the corresponding upper Riemann sum is 1.
- On the other hand, if we pick c_k to be the point giving the minimum value for f on $[x_{k-1}, x_k]$, then the corresponding lower Riemann sum is 0.
- Since the limit depends on the choices of c_k, the function f is not integrable.

Properties of Definite Integrals

The following seven properties of integrals will be useful in the process of computing integrals. We will use them repeatedly to simplify our calculations.

Theorem (Properties of Definite Integrals)

Let f and g be integrable. Then

1. *Order of Integration:* $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$
2. *Zero Width Interval:* $\int_a^a f(x)\,dx = 0$
3. *Constant Multiple:* $\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$
4. *Sum and Difference:* $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

Properties of Definite Integrals

Theorem (Properties of Definite Integrals)

5. *Additivity:* $\int_a^b f(x)\, dx + \int_b^c f(x)\, dx = \int_a^c f(x)\, dx$
6. *Max-Min Inequality: If f has maximum value max f and minimum value min f on $[a, b]$, then*

$$\min\ f.(b-a) \leq \int_a^b f(x)\, dx \leq \max\ f.(b-a).$$

7. *Domination:* $f(x) \geq g(x)$ *on* $[a, b] \Rightarrow \int_a^b f(x)\, dx \geq \int_a^b g(x)\, dx.$

$$f(x) \geq 0 \text{ on } [a, b] \Rightarrow \int_a^b f(x)\, dx \geq 0 \text{ (special case).}$$

Proof of (1). In defining $\int_b^a f(x)\,dx$ as a limit of sums $\sum_{k=1}^n f(c_k)\Delta x_k$, we moved from left to right across the interval $[a, b]$.

We now move right to left, starting from $x_0 = b$ and ending at $x_n = a$. Each Δx_k in the Riemann sum would change its sign, with $x_k - x_{k-1}$ now negative instead of positive.

With the same choices of c_k in each subinterval, the sign of any Riemann sum would change, as would the sign of the limit, the integral $\int_a^b f(x)\,dx$. This is led to give a meaning of **integrating backward** and is defined as

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

Proof of (2). Another extension of the integral is to an interval of zero width, when $a = b$. Since $f(c_k)\Delta x_k$ is zero when the interval of zero width Δx_k, we define

$$\int_a^a f(x)\,dx = 0.$$

Proof of (6). For every partition of $[a, b]$ and for every choice of the points c_k,

$$\begin{aligned} \min f.(b-a) &= \min f.\sum_{k=1}^{n} \Delta x_k \\ &= \sum_{k=1}^{n} \min f.\Delta x_k \\ &\leq \sum_{k=1}^{n} f(c_k).\Delta x_k \\ &\leq \sum_{k=1}^{n} \max f.\Delta x_k \\ &= \max f.\sum_{k=1}^{n} \Delta x_k \\ &= \max f.(b-a). \end{aligned}$$

In short, all Riemann sums for f on $[a, b]$ satisfy the inequality

$$\min f.(b-a) \leq \sum_{k=1}^{n} f(c_k)\Delta x_k \leq \max f.(b-a).$$

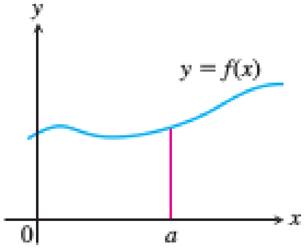
Hence their limit, the integral, does satisfy the inequality

$$\min f.(b-a) \leq \int_a^b f(x)\, dx \leq \max f.(b-a).$$

This completes the proof.

Rules 2 through 7 have geometric interpretations, shown in the following figures. The graphs in these figures are of positive functions, but the rules apply to general integrable functions.

Zero Width Interval

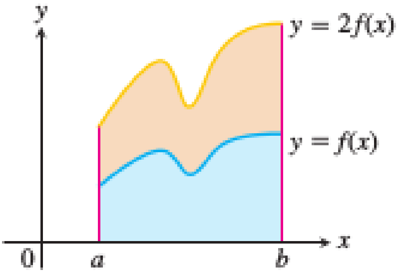


Zero Width Interval:

$$\int_a^a f(x)\,dx = 0\,.$$

(The area over a point is 0.)

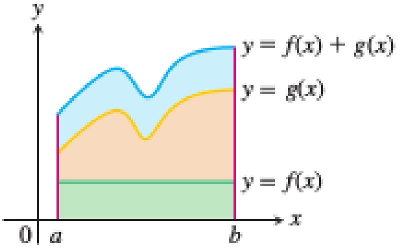
Constant Multiple



Constant Multiple:

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx.$$

Sum

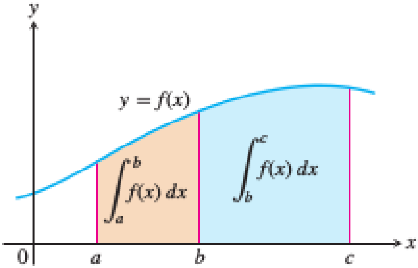


Sum:

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

(Areas add)

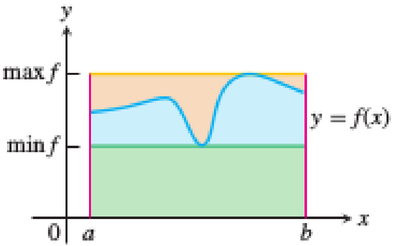
Additive for Definite Integrals



Additivity for definite integrals:

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

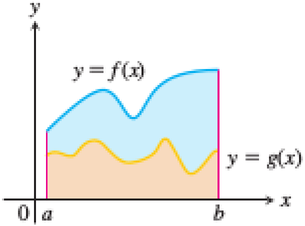
Max-Min Inequality



Max-Min Inequality:

$$\min f \cdot (b-a) \le \int_a^b f(x)\,dx \le \max f \cdot (b-a)$$

Domination



Domination:

$$f(x) \geq g(x) \text{ on } [a, b] \Rightarrow \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$

Area under the graph of a nonnegative function

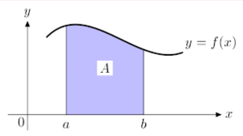
The idea of approximating a region by increasingly many rectangles, gives precise the notion of the area of a region with curved boundary.

The area under the graph of a nonnegative continuous function is defined to be a definite integral.

Definition (Area under a curve as a definite integral)

If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the area under the curve $y = f(x)$ over $[a, b]$ *is the integral of f from a to b,*

$$A = \int_a^b f(x)\, dx.$$

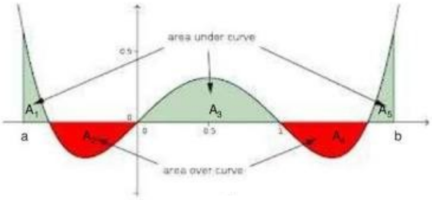


If a function f is integrable over a closed interval $[a, b]$, then

$$\int_a^b f(x)\ dx$$

is

the sum of areas under the curve $y = f(x)$ in subintervals of $[a, b]$ in which f is nonnegative — **the sum of areas over the curve $y = f(x)$ in subintervals of $[a, b]$ in which f is negative**.



Limits of Upper Sums

Exercises

2. *For the functions in the following exercises find a formula for the upper sum obtained by dividing the interval* $[a, b]$ *into* n *equal subintervals.*
   Then take a limit of these sums as $n \to \infty$ *to calculate the area under the curve over* $[a, b]$.
   (a) $f(x) = 3x^2$ *over the interval* $[0, 1]$
   (b) $f(x) = x + x^2$ *over the interval* $[0, 1]$.
3. *Express the limit*

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} \sqrt{4 - c_k^2}\ \Delta x_k$$

   where P *is a partition of* $[0, 1]$ *as a definite integral.*

Solution

2. (a) We use right endpoints to obtain upper sums.
   An upper sum is

$$\sum_{i=1}^{n} 3x_i^2 \frac{1}{n} = \sum_{i=1}^{n} 3\left(\frac{i}{n}\right)^2\left(\frac{1}{n}\right) = \frac{3}{n^3}\frac{n(n+1)(2n+1)}{6}.$$

   Thus

$$\texttt{Area} = \lim_{n\to\infty}\sum_{i=1}^{n} 3x_i^2\left(\frac{1}{n}\right) = 1.$$

   (b) We use right endpoints to obtain upper sums. Thus

$$\texttt{Area} = \lim_{n\to\infty}\sum_{i=1}^{n}(x_i + x_i^2)\left(\frac{1}{n}\right) = \frac{5}{6}.$$

Solution

3. We can express the limit

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} \sqrt{4 - c_k^2} \;\; \Delta x_k$$

as a definite integral

$$\int_0^1 \sqrt{4 - x^2} \;\; dx$$

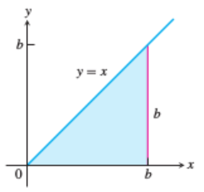
where P is a partition of $[0, 1]$.

Exercises

4. *Compute*

$$\int_0^b x \, dx$$

by limit of Riemann sums and verify that the value is the area A *under* $y = x$ *over the interval* $[0, b]$, $b > 0$.



Generalize to integrate $f(x) = x$ *over any closed interval* $[a, b]$, $0 < a < b$.

Exercises

5. *Using Riemann sums find a rule (formula) for integrating*

$$f(x) = x^3$$

   over $[a, b]$, $a < b$.

6. *Graph the integrand and use areas to evaluate the integral*

$$\int_{-1}^{1} (1 + \sqrt{1 - x^2})\, dx.$$

7. *Using* ε-δ *definition, prove that every constant function on* $[a, b]$ *is (Riemann) integrable.*

Exercises

8. *Suppose that f and g are integrable and that*

$$\int_1^2 f(x)\,dx = -4, \quad \int_1^5 f(x)\,dx = 6, \quad \int_1^5 g(x)\,dx = 8.$$

Then find

(a) $\int_2^2 g(x)\,dx$

(b) $\int_5^1 g(x)\,dx$

(c) $\int_1^2 3f(x)\,dx$

(d) $\int_2^5 f(x)\,dx$

(e) $\int_1^5 [f(x) - g(x)]\,dx$

(f) $\int_1^5 [4f(x) - g(x)]\,dx$

Solution

8. Given that f and g are integrable and that

$$\int_1^2 f(x)\ dx = -4, \quad \int_1^5 f(x)\ dx = 6, \quad \int_1^5 g(x)\ dx = 8.$$

(a) $\int_2^2 g(x)\ dx = 0$

(b) $\int_5^1 g(x)\ dx = -8$

(c) $\int_1^2 3f(x)\ dx = -12$

(d) $\int_2^5 f(x)\ dx = 10$

(e) $\int_1^5 [f(x) - g(x)]\ dx = -2$

(f) $\int_1^5 [4f(x) - g(x)]\ dx = 16$

Average Value of Continuous Functions

We now see that how to find the average value of a nonnegative continuous function f over an interval $[a, b]$, leading us to define its average.

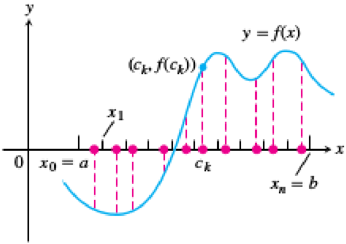
A continuous function f on $[a, b]$ may have infinitely many values, but we can still sample them in an orderly way.

We divide $[a, b]$ into n subintervals of equal width

$$\Delta x = \frac{(b - a)}{n}$$

and evaluate f at a point c_k in each.

Average Value of Continuous Functions



The average of the n sampled values is

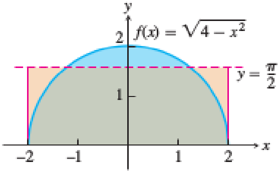
$$
\begin{aligned}
\frac{f(c_1) + f(c_2) + \cdots + f(c_n)}{n} &= \frac{1}{n} \sum_{k=1}^{n} f(c_k) \\
&= \frac{\Delta x}{b-a} \sum_{k=1}^{n} f(c_k) \\
&= \frac{1}{b-a} \sum_{k=1}^{n} f(c_k)\Delta x.
\end{aligned}
$$

The average is obtained by dividing a Riemann sum for f on $[a, b]$ by $(b - a)$. As we increase the size of the sample and let the norm of the partition approach zero, the average approaches

$$\frac{1}{b-a}\int_a^b f(x)\, dx.$$

Example

The average value of $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$ is $\pi/2$.

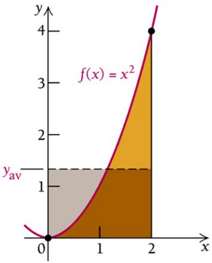


Average Value of Continuous Functions

Definition (The average or mean value of a function)

If f is integrable on $[a, b]$*, then its* average value on $[a, b]$*, also called its* mean value*, is*

$$av(f) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$



The average value of $f(x) = x^2$ on $[0, 2]$ is $\frac{4}{3}$.

Note that $\text{av}(f)\,.\,(b-a) = \int_a^b f(x)\,dx.$

In this example,

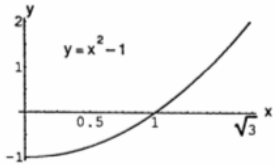
$$\frac{4}{3} \times 2 = \int_0^2 x^2\,dx.$$

Exercises

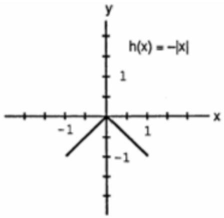
9. *In the following exercises, graph the function and find its average value over the given interval.*
   (a) $f(x) = x^2 - 1$ *on* $[0, \sqrt{3}]$.
   (b) $h(x) = -|x|$ *on each of the intervals* $[-1, 0], [0, 1]$ *and* $[-1, 1]$.

Solution

9. (a) The average value of $f(x) = x^2 - 1$ over $[0, \sqrt{3}]$ is 0.



(b) The average value of $h(x) = -|x|$ is $-\frac{1}{2}$ over the intervals $[-1, 0], [0, 1]$ and $[-1, 1]$.



Exercises

10. *What values of a and b maximize the value of* $\int_a^b (x - x^2)\, dx$?

11. *Show that the value of* $\int_0^1 \sin(x^2)\, dx$ *cannot possibly be 2.*

Solution

10. To maximize the value of the integral $\int_a^b (x - x^2)\, dx$, we should look for a bigger interval in which the integrand $x - x^2$ is non-negative. That is, to find where $x - x^2 \geq 0$. We first find x satisfying $x - x^2 = 0$. We get $x = 0$ and $x = 1$. When $0 < x < 1$, $x - x^2$ is non-negative. **Hence $a = 0$ and $b = 1$ maximize the integral.**
11. Since $-1 \leq \sin(x^2) \leq 1$, for all x,

$$(1 - 0)(-1) \leq \int_0^1 \sin(x^2)\, dx \leq (1 - 0)(1).$$

As $\int_0^1 \sin(x^2)\, dx \leq 1$, we can say that

$$\int_0^1 \sin(x^2)$$

cannot possibly be 2.

Exercises

12. *Use the Max-Min Inequality to find upper and lower bounds for the value of*

$$\int_0^1 \frac{1}{1+x^2}\, dx.$$

13. *Show that the value of* $\int_0^1 \sqrt{1+\cos x}\, dx$ *is less than or equal to* $\sqrt{2}$.

14. *Let* f *be a continuos function. Express*

$$\lim_{n\to\infty} \frac{1}{n}\left[f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \cdots + f\left(\frac{n}{n}\right)\right]$$

as a definite integral. Use the result to evaluate

$$\lim_{n\to\infty} \frac{1}{n}\left(\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \sin\frac{3\pi}{n} + \cdots + \sin\frac{n\pi}{n}\right).$$

Exercises

15. *The inequality*

$$\sec x \geq 1 + \frac{x^2}{2}$$

holds on $(-\pi/2, \pi/2)$. *Use it to find a lower bound for the value of* $\displaystyle\int_0^1 \sec x \; dx$.

16. *It would be nice if average values of integrable functions obeyed the following rules on an interval* $[a, b]$.

    (a) $av(f+g) = av(f) + av(g)$

    (b) $av(kf) = k \; av(f)$ *(any number* k*)*

    (c) $av(f) \leq av(g)$ *if* $f(x) \leq g(x)$ *on* $[a, b]$.

    Do these rules ever hold ? Give reasons for your answers.

Exercises

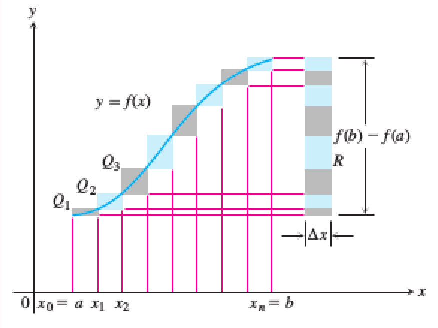
17. *Suppose the graph of a continuous function $f(x)$ rises steadily as x moves from left to right across an interval $[a, b]$. Let P be a partition of $[a, b]$ into n subintervals of length*

$$\Delta x = (b - a)/n.$$

Show by referring to the accompanying figure that the difference between the upper and lower sums for f on this partition can be represented graphically as the area of a rectangle R whose dimensions are

$$[f(b) - f(a)]$$

by Δx.



Exercises

18. *Use the formula*

$$\sin h + \sin 2h + \sin 3h + \cdots + \sin mh = \frac{\cos(h/2) - \cos((m + (1/2))h)}{2\sin(h/2)}$$

to find the area under the curve $y = \sin x$ *from* $x = 0$ *to* $x = \pi/2$ *in two steps:*

   (a) *Partition the interval* $[0, \pi/2]$ *into* n *subintervals of equal length and calculate the corresponding uper sum* U*; then*

   (b) *Find the limit of* U *as* $n \to \infty$ *as* $\Delta x = (b - a)/n \to 0$.

19. *If you average* 30 *mi/h on a* 150*-mi trip and then return over the same* 150 *mi at the rate of* 50 *mi/h, what is your average speed for the trip? Give reasons for your answer.*

Fundamental Theorem of Calculus

We now see the **Fundamental Theorem of Calculus**, which is the **central theorem of integral calculus**.

It connects integration and differentiation, enabling us to compute integrals using an antiderivative of the integrand function rather than by taking limits of Riemann sums.

Leibniz and Newton exploited this relationship and started mathematical developments that fueled the scientific revolution for the next 200 years.

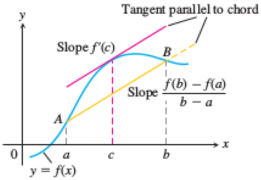
Recall

Theorem (The Mean Value Theorem)

If f is continuous on a closed interval $[a, b]$ and f is differentiable on the interval's interior (a, b). Then there is at least one point c in (a, b) at which

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Geometrically, there is a point where the tangent is parallel to the chord joining $(a, f(a))$ and $(b, f(b))$.



Recall

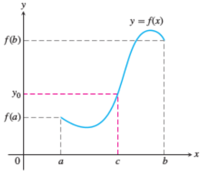
Theorem (The Intermediate Value Theorem for Continuous Functions)

A function $y = f(x)$ that is continuous on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$.
In other words, if y_0 is any value between $f(a)$ and $f(b)$, then

$$y_0 = f(c)$$

for some c in $[a, b]$.

Geometrically, the Intermediate Value Theorem says that any horizontal line $y = y_0$ crossing the y-axis between the numbers $f(a)$ and $f(b)$ will cross the curve $y = f(x)$ at least once over the interval $[a, b]$.

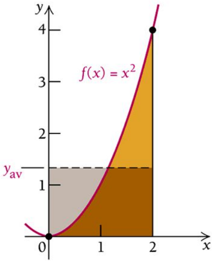


The Mean Value Theorem for Definite Integrals

First we present the **integral version of the Mean Value Theorem**, which is another **important theorem of integral calculus** and used to prove the Fundamental Theorem.

The Mean Value Theorem for Definite Integrals asserts that the average value is always taken on at least once by the function f in the interval.

Geometrically, the Mean Value Theorem says that there is a number c in $[a, b]$ such that the rectangle with height equal to the average value $f(c)$ of the function and base width $b - a$ has exactly the same area as the region beneath the graph of f from a to b.



Theorem (The Mean Value Theorem for Definite Integrals)

If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$f(c) = \frac{1}{b-a} \int_a^b f(x) \ dx.$$

Proof of the theorem : If we divide both sides of the Max-Min Inequality by $(b-a)$, we obtain

$$\min f \leq \frac{1}{b-a} \int_a^b f(x) \ dx \leq \max f.$$

Since f is continuous, the Intermediate Value Theorem for Continuous Functions says that f must assume every value between $\min f$ and $\max f$. It must therefore assume the value

$$\frac{1}{b-a} \int_a^b f(x) \ dx$$

at some point c in $[a, b]$.

Exercises

1. *Show that if* f *is continuous on* $[a, b]$, $a \neq b$, *and if*

$$\int_a^b f(x) \ dx = 0,$$

then $f(x) = 0$ *at least once in* $[a, b]$.

2. *Determine the number* c *that satisfies the Mean Value Theorem for Definite integrals for the function*

$$f(x) = x^2 + 3x + 2$$

on $[1, 4]$.

Solution

1. The Mean Value Theorem for Definite Integrals says that if f is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$f(c) = \frac{1}{b-a}\int_a^b f(x) \; dx. \tag{1}$$

Given that f is continuous on $[a, b]$, $a \neq b$, and

$$\int_a^b f(x) \; dx = 0.$$

From the equation (1), we see that $f(c) = 0$.

Solution

2. Note that the function is a polynomial and so is continuous on the given interval. Hence we can use the Mean Value Theorem for Definite Integrals. By the Mean Value Theorem for Definite Integrals, there is some c in $[1, 4]$ such that

$$f(c) = \frac{1}{3} \int_1^4 (x^2 + 3x + 2) \ \ dx.$$

So, we get $3(c^2 + 3c + 2) = \frac{99}{2}$, which implies that $3c^2 + 9c - \frac{87}{2}$, and thus $c = \frac{-3 \pm \sqrt{67}}{2}$.
The number $\frac{-3-\sqrt{67}}{2}$ is not in the interval $[1, 4]$.
Thus $\frac{-3+\sqrt{67}}{2}$ is the required one.

Theorem (Fundamental Theorem of Calculus - Part 1)

If f is continuous on $[a, b]$ then

$$F(x) = \int_a^x f(t)\, dt$$

is continuous on $[a, b]$ and differentiable on (a, b) and its derivative is $f(x)$. That is,

$$F'(x) = \frac{d}{dx} \int_a^x f(t)\, dt = f(x).$$

Proof of the theorem : If $f(t)$ is an integrable function over a finite integral I, then the integral from any fixed number $a \in I$ to another number $x \in I$ defines a new function F whose value at x is

$$F(x) = \int_a^x f(t)\, dt.$$

For $x, x+h \in (a,b)$, we have

$$F(x+h) - F(x) = \int_a^{x+h} f(t)\ dt - \int_a^x f(t)\ dt = \int_x^{x+h} f(t)\ dt$$

by additivity rule for integrals.

Hence

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h} \int_x^{x+h} f(t)\ dt. \tag{2}$$

By the Mean Value Theorem for Definite Integrals, there is some c in this interval such that

$$\frac{1}{h} \int_x^{x+h} f(t)\ dt = f(c).$$

As $h \to 0$, $x+h$ approaches x, forcing c to approach x also.

Since f is continuous at x, $f(c)$ approaches $f(x)$ as $h \to 0$. Hence

$$\begin{aligned} \frac{dF}{dx} &= \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\ dt \\ &= \lim_{h\to 0} f(c) = f(x). \end{aligned}$$

If $x = a$ or b, then the limit of Equation (2) is interpreted as a one-sided limit with $h \to 0^+$ or $h \to 0^-$, respectively.

This shows that F is continuous for every point of $[a, b]$.

Fundamental Theorem of Calculus - Part 2 (The Evaluation Theorem)

Theorem

If f is continuous at every point of $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x) \quad dx = F(b) - F(a).$$

Proof of the theorem : Part 1 of the Fundamental Theorem tells us that an antiderivative of f exists, namely

$$G(x) = \int_a^x f(t)\, dt.$$

Thus, if F is any antiderivative of f, then

$$F(x) = G(x) + C$$

for some constant C for $a < x < b$.

Since both F and G are continuous on $[a, b]$, we see that

$$F(x) = G(x) + C$$

also holds when $x = a$ and $x = b$ by taking one-sided limits (as $x \to a^+$ and $x \to b_-$).

Evaluating $F(b) - F(a)$, we have

$$\begin{aligned} F(b) - F(a) &= [G(b) + C] - [G(a) + C] \\ &= G(b) - G(a) \\ &= \int_a^b f(t)\, dt - \int_a^a f(t)\, dt \\ &= \int_a^b f(t)\, dt - 0 \\ &= \int_a^b f(t)\, dt. \end{aligned}$$

Evaluation Theorem

The Evaluation Theorem is important because it says that to calculate the definite integral of f over an interval $[a, b]$ we need do only two things:

(a) Find an antiderivative F of f, and

(b) Calculate the number $F(b) - F(a)$, which is equal to $\int_a^b f(x) \; dx$.

This process is much easier then using a Riemann sum computation.

The power of the theorem follows from the realization that the definite integral, which is defined by a complicated process involving all of the values of the function f over $[a, b]$, can be found by knowing the values of **any** antiderivative F at only the two endpoints a and b.

The usual notation for the difference $F(b) - F(a)$ is

$$F(x)\Big]_a^b \quad \text{or} \quad \Big[F(x)\Big]_a^b$$

depending on whether F has one or more terms.

Example

We calculate several definite integrals using the Evaluation Theorem, rather than by taking limits of Riemann sums.

$$\begin{aligned}\int_0^{\pi} \cos x \quad dx &= \sin x\Big]_0^{\pi} \\ &= \sin\pi - \sin 0 \\ &= 0 - 0 \\ &= 0.\end{aligned}$$

$$\frac{d}{dx}\sin x = \cos x$$

The Integral of a Rate

We can interpret Part 2 of the Fundamental Theorem in another way. If F is any antiderivative of f, then $F' = f$. The equation in the theorem can then be rewritten as

$$\int_a^b F'(x) \quad dx = F(b) - F(a).$$

Now $F'(x)$ represents the rate of change of the function $F(x)$ with respect to x, so the integral of F' is just the **net change** in F as x changes from a to b. Formally, we have the following result.

Theorem (The Net Change Theorem)

The net change in a function $F(x)$ over an interval $a \leq x \leq b$ is the integral of its rate of change :

$$F(b) - F(a) = \int_a^b F'(x) \ \ dx.$$

Displacement over the time interval

If an object with position function $s(t)$ moves along a coordinate line, its velocity is $v(t) = s'(t)$. Net Change Theorem says that

$$\int_{t_1}^{t_2} v(t)\ dt = s(t_2) - s(t_1),$$

so the integral of velocity is the **displacement** over the time interval

$$t_1 \leq t \leq t_2.$$

On the other hand, the integral of the speed

$$|v(t)|$$

is the **total distance traveled** over the time interval.

Displacement over the time interval

As

$$F(b) = F(a) + \int_a^b F'(x) \quad dx$$

we see that the Net Change Theorem also says that the final value of a function $F(x)$ over an interval $[a, b]$ equals its initial value $F(a)$ plus its net change over the interval.

So if $v(t)$ represents the velocity function of an object moving along a coordinate line, this means the the object's final position $s(t_2)$ over a time interval

$$t_1 \leq t \leq t_2$$

is its initial position $s(t_1)$ plus its net change in position along the line.

The Relationship between Integration and Differentiation

The conclusions of the Fundamental Theorem tell us several things. The equation $F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$ can be rewritten as

$$\frac{d}{dx}\int_a^x f(t)\,dt = \frac{dF}{dx} = f(x),$$

which says that **if we first integrate the function f and then differentiate the result, we get the function f back again.**

Likewise, the equation

$$\int_a^x \frac{dF}{dt} f(t)\,dt = \int_a^x f(t)\,dt = F(x) - F(a)$$

says that **if we first differentiate the function F and then integrate the result, we get the function F back** (adjusted by an integration constant).

The Relationship between Integration and Differentiation

In a sense, **the processes of integration and differentiation are "inverses" of each other.**

The Fundamental Theorem also says that **every continuous function** f **has an antiderivative** F**.**

And it says that the differential equation

$$\frac{dy}{dx} = f(x)$$

has a solution (namely, the function $y = F(x)$) for every continuous function f.

Total Area

To compute the area of the region by the graph of a function $y = f(x)$ and the x-axis requires more care when the function takes on both positive and negative values.

We must be careful to break up the interval $[a, b]$ into subintervals on which the function doesn't change sign.

Otherwise we might get cancellation between positive and negative signed areas, leading to an incorrect total.

The correct total area is obtained by adding the absolute value of the definite integral over each subinterval where $f(x)$ does not change sign.

The term "area" will be taken to mean **total area**.

Summary

To find the area between the graph of $y = f(x)$ and the x-axis over the interval $[a, b]$, do the following:

1. Subdivide $[a, b]$ at the zeros of f.
2. Integrate f over each subinterval.
3. Add the absolute values of the integrals.

Exercises

3. *Evaluate the following integrals.*

   (a) $\displaystyle\int_{\pi/6}^{5\pi/6} \csc^2 x \quad dx$

   (b) $\displaystyle\int_{-\pi/3}^{-\pi/4} \left(4\sec^2 t + \frac{\pi}{t^2}\right)\, dt$

   (c) $\displaystyle\int_{-\sqrt{3}}^{\sqrt{3}} (t+1)(t^2+4)\ dt$

   (d) $\displaystyle\int_0^{\pi} \frac{1}{2}(\cos x + |\cos x|) \quad dx.$

Solution

3. (a) $\displaystyle\int_{\pi/6}^{5\pi/6} \csc^2 x \quad dx = -\cot x \Big|_{\pi/6}^{5\pi/6} = 2\sqrt{3}$

   (b) $\displaystyle\int_{-\pi/3}^{-\pi/4} \left(4\sec^2 t + \frac{\pi}{t^2}\right)\, dt = \left[4\tan t - \frac{\pi}{t}\right]\Big|_{-\pi/3}^{-\pi/4} = 4\sqrt{3} - 3$

   (c) $\displaystyle\int_{-\sqrt{3}}^{\sqrt{3}} (t+1)(t^2+4)\, dt = 10\sqrt{3}$

   (d) $\displaystyle\int_{0}^{\pi} \frac{1}{2}(\cos x + |\cos x|) \quad dx = \Big[\sin x\Big]_0^{\pi/2} = 1.$

Exercises

Find the derivatives of the following.

(a) *by evaluating the integral and differentiating the result.*

(b) *by differentiating the integral directly.*

4. $\dfrac{d}{dx}\displaystyle\int_0^{\sqrt{x}} \cos t \; dt$
5. $\dfrac{d}{dx}\displaystyle\int_1^{\sin x} 3t^2 \; dt$
6. $\dfrac{d}{d\theta}\displaystyle\int_0^{\tan\theta} \sec^2 y \; dy$
7. $\dfrac{d}{dt}\displaystyle\int_0^{t^4} \sqrt{u}\; du.$

Solution

4. (a) $\displaystyle\int_0^{\sqrt{x}} \cos t \;dt = \sin\sqrt{x}.$

   Hence $\displaystyle\frac{d}{dx}\Big(\int_0^{\sqrt{x}} \cos t \;dt\Big) = \frac{\cos\sqrt{x}}{2\sqrt{x}}$

   (b) $\displaystyle\frac{d}{dx}\Big(\int_0^{\sqrt{x}} \cos t \;dt\Big) = cos\sqrt{x}\Big(\frac{d}{dx}(\sqrt{x})\Big) = \frac{\cos\sqrt{x}}{2\sqrt{x}}$

5. (a) $\displaystyle\int_1^{\sin x} 3t^2 \;dt = \sin^3 x - 1$

   Hence $\displaystyle\frac{d}{dx}\Big(\int_1^{\sin x} 3t^2 \;dt\Big) = 3\sin^2 x \;\cos x$

   (b) $\displaystyle\frac{d}{dx}\Big(\int_1^{\sin x} 3t^2 \;dt\Big) = (3\sin^2 x)\frac{d}{dx}(\sin x) = 3\sin^2 x \;\cos x$

Exercises

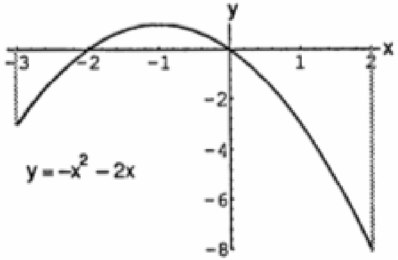
8. *Find* $\frac{dy}{dx}$ *of the following.*
   (a) $y = \int_0^x \sqrt{1+t^2}\ dt$
   (b) $y = \int_1^x \frac{1}{t}\ dt, x > 0$
   (c) $y = \int_{\sqrt{x}}^0 \sin(t^2)\ dt$
   (d) $y = \int_1^{\sin x} \frac{dt}{\sqrt{1-t^2}}, \quad |x| < \pi/2$
9. *Find the total area between the region and the x-axis*
   (a) $y = -x^2 - 2x, \quad -3 \leq x \leq 2$
   (b) $y = x^3 - 4x, \quad -2 \leq x \leq 2$
   (c) $y = x^{1/3} - x, \quad -1 \leq x \leq 8.$

Solution

8. (a) $y = \int_0^x \sqrt{1+t^2}\, dt$ implies $\frac{dy}{dx} = \sqrt{1+x^2}$

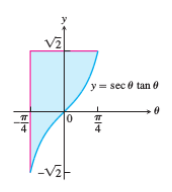
   (b) $y = \int_1^x \frac{1}{t}\, dt$ implies $\frac{dy}{dx} = \frac{1}{x}, x > 0$

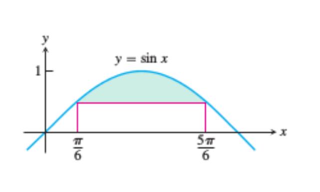
9. (a) **Limits of Integration :**
$-x^2 - 2x = 0 \implies x = 0$, or $x = -2$.
Total Area
$= -\int_{-3}^{-2}(-x^2 - 2x)\,dx + -\int_{-2}^{0}(-x^2 - 2x)\,dx - \int_{0}^{2}(-x^2 - 2x)\,dx = \frac{28}{3}$



Exercises

10. *Find the areas of the shaded regions of the following.*

(a) 

(b) 

10. (a)

$$\begin{aligned} \text{Area} &= \frac{\pi\sqrt{2}}{4} - \int_{-\pi/4}^{0} \sec\theta \tan\theta \; d\theta + \frac{\pi\sqrt{2}}{4} - \int_{0}^{\pi/4} \sec\theta\tan\theta \; d\theta \\ &= \frac{\pi\sqrt{2}}{4} + (\sqrt{2}-1) + \frac{\pi\sqrt{2}}{4} - (\sqrt{2}-1) \\ &= \frac{\pi\sqrt{2}}{2} \end{aligned}$$

(b) Area under the curve $y = \sin x$ on $[\pi/6, 5\pi/6]$ is

$$\int_{\pi/6}^{5\pi/6} \sin x \; dx = \sqrt{3}$$

Area of the shaded region is $\sqrt{3} - \frac{\pi}{3}$

Exercises

11. *Each of the following functions solves one of the initial value problems. Which function solves which problem? Give brief reasons for your answers.*

Initial value problems

(a) $y = \int_1^x \frac{1}{t}\, dt - 3$

(b) $y = \int_0^x \sec t\ dt + 4$

(c) $y = \int_{-1}^x \sec t\ dt + 4$

(d) $y = \int_\pi^x \frac{1}{t}\, dt - 3$

Solutions

(a) $\frac{dy}{dx} = \frac{1}{x}, \quad y(\pi) = -3$

(b) $y' = \sec x, \quad y(-1) = 4$

(c) $y' = \sec x, \quad y(0) = 4$

(d) $y' = \frac{1}{x}, \quad y(1) = -3.$

Exercises

12. *Express the solutions of the initial value problem in terms of integrals.*

(a) $\frac{dy}{dx} = \sec x, \quad y(2) = 3$

(b) $\frac{dy}{dx} = \sqrt{1 + x^2}, \quad y(1) = -2$

(c) $\frac{ds}{dt} = f(t), \quad s(t_0) = s_0$

(d) $\frac{dv}{dt} = g(t), \quad v(t_0) = v_0.$

Solution

12. (a) $y = \int_2^x \sec t \ dt + 3$

    (b) $y = \int_1^x \sqrt{1+t^2} \ dt - 2$

    (c) $s = \int_{t_0}^t f(x) \ dx + s_0$

    (d) $v = \int_{t_0}^t g(x) \ dx + v_0$.

Archimedes' area formula for parabolas

Exercises

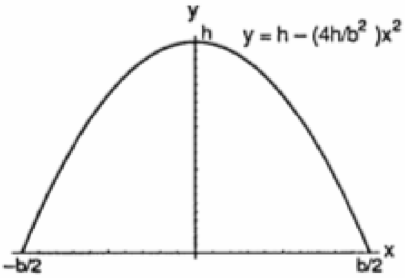
13. *Archimedes (287-212 B.C.), inventor, military engineer, physicist, and the greatest mathematician of classical times in the Western world, discovered that the area under a parabolic arch is two-thirds the base times the height. Sketch the parabolic arch*

$$y = h - (4h/b^2)x^2 \quad -b/2 \leq x \leq b/2$$

assuming that h and b are positive. Then use calculus to find the area of the region enclosed between the arch and the x-axis.

Solution

13. Area $= \int_{-b/2}^{b/2} h - (4h/b^2)x^2 \; dx = \frac{2}{3}bh.$



Exercises

14. *Consider a heavy rock blown straight up from the ground by a dynamite blast. The velocity of the rock at any time t during the motion was given as $v(t) = 160 - 32t$ ft/sec.*
    (a) *Find the displacement of the rock during the time period $0 \leq t \leq 8$.*
    (b) *Find the total distance traveled during this time period.*

15. Revenue from marginal revenue: *Suppose that a company's marginal revenue from the manufacture and sale of egg beaters is*

$$\frac{dr}{dt} = 2 - \frac{2}{(x+1)^2},$$

where r is measured in thousands of dollars and x in thousands of units. How much money should the company expect from a production run of $x = 3$ thousand egg beaters? To find out, integrate the marginal revenue from $x = 0$ to $x = 3$.

Exercises

16. *The temperature* $T(^\circ F)$ *of a room at time t minutes is given by*

$$T = 85 - 3\sqrt{25 - t} \quad \textit{for} \quad 0 \le t \le 25.$$

   (a) *Find the room's temperature when* $t = 0, t = 16$, *and* $t = 25$.
   (b) *Find the room's average temperature for* $0 \le t \le 25$.

17. *The height H (ft) of a palm tree after growing for t years is given by*

$$H = \sqrt{t + 1} + 5t^{1/3} \quad \textit{for} \quad 0 \le t \le 8.$$

   (a) *Find the tree's height when* $t = 0, t = 4$, *and* $t = 8$.
   (b) *Find the tree's average height for* $0 \le t \le 8$.

Exercises

18. *Suppose that* $\int_1^x f(t)\,dt = x^2 - 2x + 1$. *Find* $f(x)$.

19. *Find* $f(4)$ *if* $\int_0^x f(t)\,dt = x\cos \pi x$.

20. *If* $\int_0^x f(t)\,dt = x + \int_0^1 f(t)\,dt$, *then find the value of* $f(1)$.

21. *Suppose* $\int_0^x f(t)\,dt = x^2(1+x),\ x \geq 0$ *find the value of* $f(2)$.

Solution

18. $f(x) = \frac{d}{dx}\int_1^x f(t)\ dt = \frac{d}{dx}(x^2 - 2x + 1).$

    Hence
    $$f(x) = 2x - 2.$$

19. $f(x) = \frac{d}{dx}\int_0^x f(t)\ dt = \frac{d}{dx}(x\cos\pi x) = \cos\pi x - \pi x\sin\pi x.$

    Hence
    $$f(x) = \cos\pi x - \pi x\sin\pi x.$$

    Therefore $f(4) = 1$.

Exercise

If $x\sin(\pi x) = \int_0^{x^9} f(t)\, dt$, *then* $f(1)$ *and* $f(-1)$ *are, respectively,*

(a) $-\frac{\pi}{9}$ *and* $-\frac{\pi}{9}$

(b) $\frac{\pi}{9}$ *and* $-\frac{\pi}{9}$

(c) $\frac{\pi}{9}$ *and* $\frac{\pi}{9}$

(d) $-\frac{\pi}{9}$ *and* $\frac{\pi}{9}$

Exercise

Which of the following is/are true?

(a) *Let f be a continuous function on $[c, d]$. The average value of f is attained at most once in $[c, d]$.*

(b) *Let f be a function defined on $[c, d]$ and let $\{P_n\}$ be a sequence of partitions of $[c, d]$ such that $\|P_n\| \to 0$ as $n \to \infty$. If $\lim_{n\to\infty} S_{P_n}$ exists (S_{P_n} is a Riemann sum of f for the partition P_n), then f is integrable over $[c, d]$ and $\int_c^d f(x)\, dx = \lim_{n\to\infty} S_{P_n}$.*

(c) *Every discontinuous function on $[c, d]$ is not integrable.*

(d) *None of the above*

Exercise

If f' is continuous in $[a, b]$, then prove that

$$\left| \int_a^b f(t)dt \right| \leq \int_a^b |f(t)| \, dt.$$

Further, if $|f'(x)| \leq M$ in $[a, b]$, show that

$$|f(x)| \leq |f(a)| + M(x - a),$$

for all $x \in [a, b]$.

Table of Integrals : Basic Forms

- $\int x^n \; dx = \frac{1}{n+1} x^{n+1}, \; n \neq -1$
- $\int \frac{1}{x} \; dx = \ln |x|$
- $\int udv = uv - \int vdu$
- $\int \frac{1}{ax+b} \; dx = \frac{1}{a} \ln |ax + b|$

Integrals of Rational Functions

- $\int \frac{1}{(x+a)^2} \; dx = -\frac{1}{x+a}$
- $\int (x+a)^n \; dx = \frac{(x+a)^{n+1}}{n+1}, n \neq -1$
- $\int x(x+a)^n \; dx = \frac{(x+a)^{n+1}((n+1)x-a)}{(n+1)(n+2)}$
- $\int \frac{1}{a^2+x^2} \; dx = \frac{1}{a} \tan^{-1} \frac{x}{a}$
- $\int \frac{x}{a^2+x^2} \; dx = \frac{1}{2} \ln |a^2+x^2|$
- $\int \frac{x^2}{a^2+x^2} \; dx = x - a \tan^{-1} \frac{x}{a}$

Integrals of Rational Functions

- $\displaystyle\int \frac{x^3}{a^2+x^2}\, dx = \frac{1}{2}x^2 - \frac{1}{2}a^2 \ln|a^2+x^2|$
- $\displaystyle\int \frac{1}{ax^2+bx+c}\, dx = \frac{2}{\sqrt{4ac-b^2}} \tan^{-1} \frac{2ax+b}{\sqrt{4ac-b^2}}$
- $\displaystyle\int \frac{1}{(x+a)(x+b)}\, dx = \frac{1}{b-a} \ln \frac{a+x}{b+x}, \;\; a \neq b$
- $\displaystyle\int \frac{1}{1+x^2}\, dx = \tan^{-1} x$
- $\displaystyle\int \frac{x}{(x+a)^2}\, dx = \frac{a}{a+x} + \ln|a+x|$
- $\displaystyle\int \frac{x}{ax^2+bx+c}\, dx =$
  $\displaystyle\frac{1}{2a} \ln|ax^2+bx+c| - \frac{b}{a\sqrt{4ac-b^2}} \tan^{-1} \frac{2ax+b}{\sqrt{4ac-b^2}}$

Integrals with Roots

- $\int \sqrt{x-a} \quad dx = \frac{2}{3}(x-a)^{3/2}$
- $\int \frac{1}{\sqrt{x \pm a}} \quad dx = 2\sqrt{x \pm a}$
- $\int \sqrt{ax+b} \quad dx = \left(\frac{2b}{3a} + \frac{2x}{3}\right)\sqrt{ax+b}$
- $\int (ax+b)^{3/2} \quad dx = \frac{2}{5a}(ax+b)^{5/2}$
- $\int \sqrt{\frac{x}{a-x}} \quad dx = -\sqrt{x(a-x)} - a\tan^{-1}\frac{\sqrt{x(a-x)}}{x-a}$
- $\int \sqrt{\frac{x}{a+x}} \quad dx = \sqrt{x(a+x)} - a\ln\left[\sqrt{x} + \sqrt{x+a}\right]$
- $\int x\sqrt{ax+b} \quad dx = \frac{2}{15a^2}(-2b^2 + abx + 3a^2x^2)\sqrt{ax+b}$

Integrals with Roots

- $\int \sqrt{a^2 - x^2} \quad dx = \frac{1}{2}x\sqrt{a^2 - x^2} + \frac{1}{2}a^2 \tan^{-1} \frac{x}{\sqrt{a^2 - x^2}}$
- $\int x\sqrt{x^2 \pm a^2} \; dx = \frac{1}{3}\left(x^2 \pm a^2\right)^{3/2}$
- $\int \frac{1}{\sqrt{x^2 \pm a^2}} \quad dx = \ln\left|x + \sqrt{x^2 \pm a^2}\right|$
- $\int \frac{1}{\sqrt{a^2 - x^2}} \quad dx = \sin^{-1} \frac{x}{a}$
- $\int \frac{x}{\sqrt{x^2 \pm a^2}} \quad dx = \sqrt{x^2 \pm a^2}$
- $\int \frac{x}{\sqrt{a^2 - x^2}} \quad dx = -\sqrt{a^2 - x^2}$
- $\int \frac{x^2}{\sqrt{x^2 \pm a^2}} \quad dx = \frac{1}{2}x\sqrt{x^2 \pm a^2} \mp \frac{1}{2}a^2 \ln\left|x + \sqrt{x^2 \pm a^2}\right|$

Integrals with Logarithms

- $\int \ln ax \quad dx = x \ln ax - x$
- $\int x \ln x \quad dx = \frac{1}{2}x^2 \ln x - \frac{x^2}{4}$
- $\int x^2 \ln x \quad dx = \frac{1}{3}x^3 \ln x - \frac{x^3}{9}$
- $\int x^n \ln x \quad dx = x^{n+1}\left(\frac{\ln x}{n+1} - \frac{1}{(n+1)^2}\right), \quad n \neq -1$
- $\int \frac{\ln ax}{x} \quad dx = \frac{1}{2}(\ln ax)^2$

Integrals with Logarithms

- $\displaystyle \int \frac{\ln x}{x^2} \quad dx = -\frac{1}{x} - \frac{\ln x}{x}$
- $\displaystyle \int \ln(ax+b) \quad dx = \left(x + \frac{b}{a}\right) \ln(ax+b) - x, a \neq 0$
- $\displaystyle \int \ln(x^2+a^2) \;\; dx = x \ln(x^2+a^2) + 2a \tan^{-1} \frac{x}{a} - 2x$
- $\displaystyle \int \ln(x^2-a^2) \;\; dx = x \ln(x^2-a^2) + a \ln \frac{x+a}{x-a} - 2x$

Integrals with Exponentials

- $\int e^{ax} \quad dx = \frac{1}{a} e^{ax}$
- $\int xe^{x} \quad dx = (x-1)e^{x}$
- $\int xe^{ax} \quad dx = \left(\frac{x}{a} - \frac{1}{a^2}\right) e^{ax}$
- $\int x^2 e^{x} \quad dx = \left(x^2 - 2x + 2\right) e^{x}$
- $\int x^2 e^{ax} \quad dx = \left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right) e^{ax}$
- $\int x^3 e^{x} \quad dx = \left(x^3 - 3x^2 + 6x - 6\right) e^{x}$
- $\int x^n e^{ax} \quad dx = \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax} \, \mathrm{d}x$

Integrals with Trigonometric Functions

- $$\int \sin ax \quad dx = -\frac{1}{a}\cos ax$$
- $$\int \sin^2 ax \quad dx = \frac{x}{2} - \frac{\sin 2ax}{4a}$$
- $$\int \sin^3 ax \quad dx = -\frac{3\cos ax}{4a} + \frac{\cos 3ax}{12a}$$
- $$\int \cos ax \quad dx = \frac{1}{a}\sin ax$$
- $$\int \cos^2 ax \quad dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$$
- $$\int \cos^3 ax \quad dx = \frac{3\sin ax}{4a} + \frac{\sin 3ax}{12a}$$
- $$\int \cos x \sin x \quad dx = \frac{1}{2}\sin^2 x + c_1 = -\frac{1}{2}\cos^2 x + c_2 = -\frac{1}{4}\cos 2x + c_3$$
- $$\int \cos ax \sin bx \quad dx = \frac{\cos[(a-b)x]}{2(a-b)} - \frac{\cos[(a+b)x]}{2(a+b)}, a \neq b$$

Integrals with Trigonometric Functions

- $$\int \sin^2 ax \cos bx \quad dx = -\frac{\sin[(2a-b)x]}{4(2a-b)} + \frac{\sin bx}{2b} - \frac{\sin[(2a+b)x]}{4(2a+b)}$$
- $$\int \sin^2 x \cos x \quad dx = \frac{1}{3}\sin^3 x$$
- $$\int \cos^2 ax \sin bx \quad dx = \frac{\cos[(2a-b)x]}{4(2a-b)} - \frac{\cos bx}{2b} - \frac{\cos[(2a+b)x]}{4(2a+b)}$$
- $$\int \cos^2 ax \sin ax \quad dx = -\frac{1}{3a}\cos^3 ax$$
- $$\int \sin^2 ax \cos^2 ax \quad dx = \frac{x}{8} - \frac{\sin 4ax}{32a}$$
- $$\int \tan ax \quad dx = -\frac{1}{a}\ln \cos ax$$
- $$\int \tan^2 ax \quad dx = -x + \frac{1}{a}\tan ax$$

Integrals with Trigonometric Functions

- $\int \tan^3 ax \quad dx = \frac{1}{a} \ln \cos ax + \frac{1}{2a} \sec^2 ax$
- $\int \sec x \quad dx = \ln |\sec x + \tan x| = 2 \tanh^{-1} \left(\tan \frac{x}{2} \right)$
- $\int \sec^2 ax \quad dx = \frac{1}{a} \tan ax$
- $\int \sec^3 x \quad dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x|$
- $\int \sec x \tan x \quad dx = \sec x$
- $\int \sec^2 x \tan x \quad dx = \frac{1}{2} \sec^2 x$
- $\int \sec^n x \tan x \quad dx = \frac{1}{n} \sec^n x, n \neq 0$

Integrals with Trigonometric Functions

- $\displaystyle\int \csc x \quad dx = \ln\left|\tan\frac{x}{2}\right| = \ln|\csc x - \cot x| + C$
- $\displaystyle\int \csc^2 ax \quad dx = -\frac{1}{a}\cot ax$
- $\displaystyle\int \csc^3 x \quad dx = -\frac{1}{2}\cot x \csc x + \frac{1}{2}\ln|\csc x - \cot x|$
- $\displaystyle\int \csc^n x \cot x \quad dx = -\frac{1}{n}\csc^n x, n \neq 0$
- $\displaystyle\int \sec x \csc x \quad dx = \ln|\tan x|$

Products of Trigonometric Functions and Monomials

- $\int x \cos x \quad dx = \cos x + x \sin x$
- $\int x \cos ax \quad dx = \frac{1}{a^2} \cos ax + \frac{x}{a} \sin ax$
- $\int x^2 \cos x \quad dx = 2x \cos x + \left(x^2 - 2\right) \sin x$
- $\int x^2 \cos ax \quad dx = \frac{2x \cos ax}{a^2} + \frac{a^2x^2 - 2}{a^3} \sin ax$
- $\int x \sin x \quad dx = -x \cos x + \sin x$
- $\int x \sin ax \quad dx = -\frac{x \cos ax}{a} + \frac{\sin ax}{a^2}$

Products of Trigonometric Functions and Monomials

- $\int x^2 \sin x \quad dx = \left(2 - x^2\right) \cos x + 2x \sin x$
- $\int x^2 \sin ax \quad dx = \frac{2 - a^2x^2}{a^3} \cos ax + \frac{2x \sin ax}{a^2}$
- $\int x \cos^2 x \quad dx = \frac{x^2}{4} + \frac{1}{8} \cos 2x + \frac{1}{4} x \sin 2x$
- $\int x \sin^2 x \quad dx = \frac{x^2}{4} - \frac{1}{8} \cos 2x - \frac{1}{4} x \sin 2x$
- $\int x \tan^2 x \quad dx = -\frac{x^2}{2} + \ln \cos x + x \tan x$
- $\int x \sec^2 x \quad dx = \ln \cos x + x \tan x$

Products of Trigonometric Functions and Exponentials

- $\int e^x \sin x \quad dx = \frac{1}{2} e^x (\sin x - \cos x)$
- $\int e^{bx} \sin ax \quad dx = \frac{1}{a^2 + b^2} e^{bx} (b \sin ax - a \cos ax)$
- $\int e^x \cos x \quad dx = \frac{1}{2} e^x (\sin x + \cos x)$
- $\int e^{bx} \cos ax \quad dx = \frac{1}{a^2 + b^2} e^{bx} (a \sin ax + b \cos ax)$
- $\int x e^x \sin x \quad dx = \frac{1}{2} e^x (\cos x - x \cos x + x \sin x)$
- $\int x e^x \cos x \quad dx = \frac{1}{2} e^x (x \cos x - \sin x + x \sin x)$

Integrals of Hyperbolic Functions

- $\displaystyle\int \cosh ax \quad dx = \frac{1}{a}\sinh ax$
- $\displaystyle\int \sinh ax \quad dx = \frac{1}{a}\cosh ax$
- $\displaystyle\int \tanh ax \quad dx = \frac{1}{a}\ln\cosh ax$
- $\displaystyle\int \cos ax \cosh bx \quad dx = \frac{1}{a^2+b^2}\left[a\sin ax\cosh bx + b\cos ax\sinh bx\right]$
- $\displaystyle\int \cos ax \sinh bx \quad dx = \frac{1}{a^2+b^2}\left[b\cos ax\cosh bx + a\sin ax\sinh bx\right]$
- $\displaystyle\int \sin ax \cosh bx \quad dx = \frac{1}{a^2+b^2}\left[-a\cos ax\cosh bx + b\sin ax\sinh bx\right]$
- $\displaystyle\int \sin ax \sinh bx \quad dx = \frac{1}{a^2+b^2}\left[b\cosh bx\sin ax - a\cos ax\sinh bx\right]$
- $\displaystyle\int \sinh ax \cosh ax \; dx = \frac{1}{4a}\left[-2ax + \sinh 2ax\right]$
- $\displaystyle\int \sinh ax \cosh bx \quad dx = \frac{1}{b^2-a^2}\left[b\cosh bx\sinh ax - a\cosh ax\sinh bx\right]$